Exercises on Randomized Complexity
CSCI 6114 Fall 2021

Joshua A. Grochow

October 28, 2021

Recall that we have a language L is in...

	BPP	RP	coRP
$x \in L \Rightarrow$	$Pr[A(x) \text{ accepts}] \geq 2/3$	$Pr[A(x) \text{ accepts}] \geq 2/3$	$Pr[A(x) \text{ accepts}] = 1$
$x \notin L \Rightarrow$	$Pr[A(x) \text{ accepts}] \leq 1/3$	$Pr[A(x) \text{ accepts}] = 0$	$Pr[A(x) \text{ accepts}] \leq 1/3$

Last time we saw that Polynomial Identity Testing is in coRP.

1. Show that $\mathsf{P}^{\mathsf{BPP}} = \mathsf{BPP}$, and then show that $\mathsf{BPP}^{\mathsf{BPP}} = \mathsf{BPP}$.

2. Show that $\mathsf{coBPP} = \mathsf{BPP}$.

3. Give an alternative characterization of BPP similar to the verifier definition of NP. Here, instead of a "witness", think of the Verifier $V(x, r)$ as taking in the input x and a random string r.

4. Show that $\mathsf{RP} \subseteq \mathsf{NP}$. Show that $\mathsf{NP} \subseteq \mathsf{BPP}$ iff $\mathsf{NP} = \mathsf{RP}$. *Hint:* BPP and RP are both closed under $\leq_m^p$, so NP is contained in one of these classes iff SAT is.

   We can now update our table above to:

$Pr(A(x) \text{ accepts})$	BPP	RP	coRP	NP	coNP	PP
$x \in L \Rightarrow$	$\geq 2/3$	$\geq 2/3$	$= 1$	> 0	$= 0$	$> 1/2$
$x \notin L \Rightarrow$	$\leq 1/3$	$= 0$	$\leq 1/3$	$= 0$	> 0	$\leq 1/2$

5. (a) Let $L \in \mathsf{BPP}$. Let L' be the poly(n)-concatenation of L, that is, a tuple $(x_1, \ldots, x_n) \in L'$ iff all $x_i \in L$. Show that $L' \in \mathsf{BPP}$.
   (b) Show that $\mathsf{NP}^{\mathsf{BPP}} = \mathsf{NP}^{\mathsf{BPP}[1]}$ where the latter means the oracle is queried only once. *Hint:* Use nondeterminism to guess the oracle answers, and use the one query at the end to verify the guesses.
   (c) Show that $\mathsf{NP}^{\mathsf{BPP}} \subseteq \mathsf{BPP}^{\mathsf{NP}}$.
   (d) Use the preceding to show that if $\mathsf{NP} \subseteq \mathsf{BPP}$ then $\mathsf{PH} \subseteq \mathsf{BPP}$. (We'll see next week that $\mathsf{BPP} \subseteq \Sigma_2\mathsf{P} \cap \Pi_2\mathsf{P}$, so in fact the latter implies that PH collapses.)

Resources

- TODO